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Killing scalar of non-linear σ-model on G/H realizing the classical exchange algebra

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ABSTRACT

The Poisson brackets for non-linear σ-models on G/H are set up on the light-like plane. A quantity which transforms irreducibly by the Killing vectors, called Killing scalar, is constructed in an arbitrary representation of G. It is shown to satisfy the classical exchange algebra.

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1. Introduction

The Yang–Baxter equation arises in a large class of exactly solvable models, such as lattice models, spin-chain system, nonlinear σ-models, conformal field theory, etc. Among them the Yang–Baxter equation for the PSU(2, 2|4) spin-chain system gained much interest in the last decade. Finding its solution leads to a discovery of string/QCD duality, namely, a relationship between the PSU(2, 2|4) spin-chain system and the $N = 4$ supersymmetric QCD [1,2].

To explain the Yang–Baxter equation and the resulting algebraic structure, we take a generic spin-system equipped with a set quantum operators, say, Ψ. We consider a tensor product chain of Ψs and exchange two of them in an adjacent position, say, $\Psi(x)$ and $\Psi(y)$. Then there may exists a R-matrix defining a quantum exchange algebra

$$R_{xy}\Psi(x) \otimes \Psi(y) = \Psi(y) \otimes \Psi(x), \tag{1.1}$$

such that it satisfies the Yang–Baxter equation

$$R_{xy}R_{xz}R_{yz} = R_{yz}R_{xz}R_{xy}. \tag{1.2}$$

Suppose the R-matrix to be a quantum deformation of a certain classical r-matrix as

$$R_{xy} = 1 + hr_{xy} + O\left(h^2\right),$$

with an infinitesimal parameter h. Then (1.1) and (1.2) respectively become a classical exchange algebra

$$\left\{\Psi(x) \overset{\otimes}{,} \Psi(y)\right\} = -hr_{xy}\Psi(x) \otimes \Psi(y), \tag{1.3}$$

and the classical Yang–Baxter equation

$$[r_{xy}, r_{xz}] + [r_{xy}, r_{yz}] + [r_{xz}, r_{yz}] = 0. \tag{1.4}$$

The quantity on the l.h.s. of (1.3) is a Poisson bracket which was substituted for the commutator $[\Psi(x) \overset{\otimes}{,} \Psi(y)]$. In the recent works [3,4] the classical exchange algebra (1.3) was shown for a classical quantity, called G-primary, in the constrained WZWN model on the coset space $G/\{S \otimes U(1)^d\}$. Namely, the G-primary was constructed out of basic fields of the model in an arbitrary representation of G. By setting up the Poisson brackets for the basic fields, the classical exchange algebra (1.3) was shown to appear with the r-matrix in an arbitrarily chosen representation of the G-primary [4]. It would be promoted to the quantum exchange algebra (1.2) by the usual quantization of the constrained WZWN model. Or in mathematics the algebraic construction of the R-matrix is known when the R-matrix exists in the Hopf algebra A in such a way that $R \in A \otimes A$ and

i. $\Delta'(a) = R\Delta(a)R^{-1}, \quad \forall a \in A,$

ii. $(\Delta \otimes 1)(R) = R_{xz}R_{yz}, \qquad (1 \otimes \Delta)(R) = R_{xz}R_{xy}.$

Here Δ is the coproduct and $\Delta' = P \circ \Delta$ with the permutation map P. The Yang–Baxter equation is derived as one of the properties of this Hopf algebra [5].

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In this Letter we show that the exchange algebra exists for the ordinary non-linear σ-model on G/H as well. But generalizing the arguments for the constrained WZWN model on $\mathrm{G}/\{\mathrm{S}\otimes\mathrm{U}(1)^d\}$ [3,4] to this case is not straightforward. To see a difference between the two cases, we look a bit more closely the arguments done in [3,4]. The fundamental Poisson bracket of the WZWN model was set up for the basic field

$$g = g_{\mathrm{L}} g_{\mathrm{H}}. \tag{1.5}$$

This basic field was obtained by reparametrizing the field of the WZWN model as $g = g_{\mathrm{L}} g_{\mathrm{H}} g_{\mathrm{R}} \in \mathrm{G}$ in accordance with the Gauß decomposition of the Lie-algebra of G under $\mathrm{H} \equiv \mathrm{S}\otimes\mathrm{U}(1)^d$

$$\{T_{\mathrm{G/H}}, T_{\mathrm{H}}\} \equiv \{T_{\mathrm{L}}, T_{\mathrm{R}}, T_{\mathrm{H}}\} \tag{1.6}$$

and constraining g_{R} to be 1. Here the decomposition of $T_{\mathrm{G/H}}$ into the sets T_{L} and T_{R} is done by the sign assignment of a $\mathrm{U}(1)_Y$ charge, which is defined by a certain linear combination of d U(1)s. T_{L} (or T_{R}) is further decomposed into subsets, each of which consists of generators in an irreducible representation under S with a definite value of the $\mathrm{U}(1)_Y$ charge. Then the G-primary which has the conformal weight 0 and linearly transforms in an arbitrary representation of G by the Killing vectors was constructed from the basic field (1.5). It satisfied the classical exchange algebra (1.3) when calculated the Poisson bracket by means of the fundamental one for (1.5). We may wonder if such a classical exchange algebra exists for the ordinary non-linear σ-model on G/H, which has a different reparametrization from (1.5). Namely, the basic field of the non-linear σ-model on G/H defined by a coset element

$$g = g_{\mathrm{G/H}}. \tag{1.7}$$

Here the subgroup H is not necessarily of the type $\mathrm{S}\otimes\mathrm{U}(1)^d$, i.e., the Gauß decomposition (1.6) does not need to be required. In this Letter we will show that the classical exchange algebra (1.3) exists also for this case. Indeed we will find a quantity satisfying the classical exchange algebra in an arbitrary representation of G, such as the G-primary for the constrained WZWN model. But the quantity is called Killing scalar this time, because the ordinary non-linear σ-model has no longer conformal symmetry and the construction differs from that of the G-primary.

The reader might think of non-linear σ-models on G/H as obtained by gauging the WZWN model under H and fixing the gauge. To be explicit, the field of the WZWN model is reparametrized in a form $g = g_{\mathrm{G/H}} g_H$. Fixing the gauge as $g_{\mathrm{H}} = 1$ we then get the basic field of the ordinary non-linear σ-model on G/H as given by (1.7). If $\mathrm{H} = \mathrm{S}\otimes\mathrm{U}(1)^d$, the field of the WZWN model may be alternatively reparametrized as $g = g_{\mathrm{L}} g_{\mathrm{H}} g_{\mathrm{R}}$. For that case the gauge-fixing yields the basic field as $g = g_{\mathrm{L}} g_{\mathrm{R}}$ instead of (1.5). For either case the gauged action has vector-like invariance. It no longer has the chiral invariance which played an essential role for constructing the G-primary in [3,4]. (See (3.1) in [6] and (10) in [4] for the gauged action with the respective invariance.) Therefore the same strategy as for the constrained WZWN model on $\mathrm{G}/\{\mathrm{S}\otimes\mathrm{U}(1)^d\}$ [3,4] does not work for the case with vector-like gauge invariance. We need to develop a new strategy for the ordinary non-linear σ-model on G/H.

So it is a central issue of this Letter to construct the Killing scalar for the ordinary non-linear σ-model, which plays the same role as the G-primary for the constrained WZWN model. It is discussed in Section 2. There we find that the Wilson line operator is necessary for the construction in a generic representation of G. It was not needed for the construction of the G primary at all. In Section 3 we discuss the Poisson structure of the ordinary non-linear σ-model on the light-like plane. The fundamental Poisson brackets are set up consistently on that plane. By means of them we show the classical exchange algebra for the Killing scalar.

2. The Killing scalar

We begin by giving a general account on the ordinary non-linear σ-model to an extent such that it is needed for arguments for this Letter. The standard reparametrization of the coset space G/H is given by the CCWZ formalism [7] in the following procedure.[1] Decompose the generators of G as

$$\{T^A\} = \{X^i, H^I\}, \tag{2.1}$$

in which H^I are generators of the homogeneous group H, while X^i broken ones. Then we consider a unitary quantity as the basic field (1.5)

$$U = e^{i\phi^1 X^1 + i\phi^2 X^2 + \cdots} \equiv e^{i\phi\cdot X}. \tag{2.2}$$

Here $\phi^1, \phi^2, \ldots$ were introduced correspondingly to the broken generators X^i and they are real coordinates reparametrizing the coset space G/H, denoted by ϕ^a. The Cartan–Maurer 1-form $U^{-1}dU$ is valued in the Lie-algebra of G as

$$U^{-1}dU = \left(e^i_a X^i + \omega^I_a H^I\right) d\phi^a. \tag{2.3}$$

This defines the vielbein e^i_a and the connection ω^I in the local frame of the coset space G/H. For an element $e^{i\epsilon^A T^A}$ with real parameters ϵ^A there exists a compensator $e^{i\rho(\phi,\epsilon)} \in \mathrm{H}$ such that

$$e^{i\epsilon^A T^A} U(\phi) = U\left(\phi'\right) e^{i\rho(\phi,\epsilon)}. \tag{2.4}$$

This defines a transformation of the coordinates $\phi^a \to \phi'^a(\phi)$. When ϵ^A are infinitesimally small, this relation defines the Killing vectors R^{Aa} as

$$\delta\phi^a = \phi'^a(\phi) - \phi^a \equiv \epsilon^A R^{Aa}(\phi) \equiv \delta^A \phi^a. \tag{2.5}$$

Requiring the Jacobi-identity for the transformation

$$\left(\left[\delta^A, \left[\delta^B, \delta^C\right]\right] - \left[\delta^B, \left[\delta^A, \delta^C\right]\right]\right)\phi^b = \left[\left[\delta^A, \delta^B\right], \delta^C\right]\phi^b,$$

we get the Lie-algebra of G

$$R^{Aa} R^{Bb}{}_{,a} - R^{Ba} R^{Ab}{}_{,a} = f^{ABC} R^{Cb}, \tag{2.6}$$

with $R^{Bb}{}_{,a} \equiv \partial R^{Bb}/\partial\phi^a$. This can be written as $\mathcal{L}_{R^A} R^{Bb} = f^{ABC} R^{Cb}$ by using the Lie-variation.

So far our arguments are irrelevant to the representation of G. Let us choose it to be an N-dimensional irreducible representation of G. Under the subgroup H let it to be decomposed into irreducible ones as

$$\boldsymbol{N} = \boldsymbol{N}^{w_1} \oplus \boldsymbol{N}^{w_2} \oplus \cdots \oplus \boldsymbol{N}^{w_{m-1}} \oplus \boldsymbol{N}^{w_m}. \tag{2.7}$$

Here $\boldsymbol{N}^{w_\mu}$, $\mu = 1, 2, \ldots, m$, denote the N_μ-dimensional representation of H with a set of weight vectors w_μ. Accordingly U is represented by an $N \times N$ matrix $\mathcal{D}(U)$ in a decomposed form as

$$\mathcal{D}(U) = \begin{pmatrix} (U)_{N_1\times N_1} & (U)_{N_1\times N_2} & \cdots & (U)_{N_1\times N_m} \\ (U)_{N_2\times N_1} & (U)_{N_2\times N_2} & \cdots & (U)_{N_2\times N_m} \\ \vdots & \vdots & \ddots & \vdots \\ (U)_{N_m\times N_1} & (U)_{N_m\times N_2} & \cdots & (U)_{N_m\times N_m} \end{pmatrix}, \tag{2.8}$$

in which $(U)_{N_\mu\times N_\nu}$ is an $N_\mu \times N_\nu$ matrix. From this we take out a set of column vectors such as

[1] The BKMU formalism is also useful formalism when the coset space admits the complex structure. It is straightforward to adapt the arguments in this section to the BKMU formalism [8].

$$\Psi = \begin{pmatrix} (U)_{N_1 \times N_\nu} \\ (U)_{N_2 \times N_\nu} \\ \vdots \\ (U)_{N_m \times N_\nu} \end{pmatrix}. \tag{2.9}$$

Then it transform as $(\boldsymbol{N}, \overline{\boldsymbol{N}}^{-w_\nu})$ by the transformation defined by (2.4), i.e.,

$$\Psi \longrightarrow \left(e^{i\epsilon^A T^A}\right)_{N\times N} \Psi \left(e^{-i\rho(\phi,\epsilon)}\right)_{N_\nu \times N_\nu}, \quad \nu = 1, 2, \ldots, m. \tag{2.10}$$

If $\boldsymbol{N}^{w_\nu}$ happens to be a singlet, then Ψ transforms as $\boldsymbol{N}$, i.e.,

$$\delta\Psi = \epsilon^A \mathcal{D}\left(T^A\right)\Psi. \tag{2.11}$$

We call this quantity Killing scalar since (2.11) can be written as

$$\mathcal{L}_{R^A}\Psi = \mathcal{D}\left(T^A\right)\Psi$$

by using the Lie-variation. For the coset space $SO(6)/SO(5)(=S^5)$ the Killing scalar transforming as the fundamental representation **6** of SO(6) was given in [9]. The decomposition of **6** under SO(5) indeed contains a singlet. But a generic representation $\boldsymbol{N}$ of G does not contain it in the decomposition (2.7). For example the adjoint representation **15** of SO(6) is decomposed as $\mathbf{10} \oplus \mathbf{5}$ under SO(5). Hence any column vector Ψ in the representation **15** is not the Killing scalar of the coset space $SO(6)/SO(5)(=S^5)$. A similar comment can be done for the coset space $SU(N)/\{U(1)\}^{N-1}$. The fundamental representation is decomposed into N components each of which transforms under $\{U(1)\}^{N-1}$ having U(1) charges designated by the weight vector w_μ, $\mu = 1, 2, \ldots, w_N$. Therefore Ψ in the fundamental representation is not the Killing scalar. However it is worth noting that the elements of $\mathcal{D}(U)$ defined by (2.8) in the fundamental representation for $SU(N)/\{U(1)\}^{N-1}$ were identical to the quantities known as the harmonic coordinates of the coset space [10]. The representation which has the decomposition containing singlets and provides us with the Killing scalar is the adjoint representation or the one obtained by its symmetric tensor product. Indeed the adjoint representation of $SU(N)$ contains $N-1$ null weight vectors. Note that the coset space $SU(N)/\{U(1)\}^{N-1}$ is a Kähler manifold. Ψ in the adjoint representation, transforming as (2.11), is nothing but the quantity known as the Killing potential in the literature. The naming "Killing scalar" is based on this observation.

For the case where the representation $\boldsymbol{N}$ does not contain any singlet in the decomposition under H, we can turn Ψ given by (2.9) to the Killing scalar in the following way. Consider the Wilson line operator

$$W(\phi, \phi_0) = P \exp \int_{\phi_0}^{\phi} d\phi^a \omega_a^I H^I.$$

Here $\omega_a^I H^I$ is the connection defined from the Cartan–Maurer 1-form as (2.3) and transforms as

$$\delta\left(\omega_a^I H^I\right)_{N_\nu \times N_\nu} = \left(\partial_a \rho(\phi, \epsilon) - \left[\omega_a^I H^I, \rho(\phi, \epsilon)\right]\right)_{N_\nu \times N_\nu}$$

by (2.4). Hence the Wilson line operator transforms as

$$W(\phi, \phi_0) \longrightarrow e^{i\rho(\phi,\epsilon)} W(\phi, \phi_0) e^{-i\rho(\phi_0,\epsilon)},$$

by (2.4). The generator of the compensator $\rho(\phi, \epsilon)$ becomes $\epsilon^I H^I$ at the origin $\phi_0 = 0$ of the coset space G/H. Let η to be a linear representation vector $e^{i\theta^I H^I}\eta_0$ with θ^I parametrizing the subgroup H, so that it transforms by the compensator at the origin as

$$\eta \equiv e^{i\theta^I H^I}\eta_0 \longrightarrow e^{i(\theta^I + \epsilon^I)H^I}\eta_0.$$

Here η_0 is a constant vector which is fixed in the representation space of H^I. Then we have

$$W(\phi, 0)\eta \longrightarrow e^{i\rho(\phi,\epsilon)} W(\phi, 0)\eta.$$

We have already known that by (2.4) the quantity (2.9) transforms as $(\boldsymbol{N}, \overline{\boldsymbol{N}}^{-w_\nu})$, i.e., (2.10). As the result the following quantity

$$\Psi(x) W(\phi, 0)\eta \tag{2.12}$$

is the Killing scalar transforming as (2.11) in any representation $\boldsymbol{N}$ of G. The point of the argument here is that the existence of a quantity transforming as $(\mathbf{1}, \boldsymbol{N}^{w_\nu})$ by (2.4) is not hypothetical, but it indeed exists as $W(\phi, 0)\eta$.

It is here opportune to discuss a relation between the vielbein e_a^i and the Killing vectors R^{Aa} given by (2.3) and (2.5) respectively. Let $\mathcal{D}(U)$ given by (2.8) to be the adjoint representation of G. As the adjoint representation is decomposed as (2.1) we have

$$\mathcal{D}(U) = \left(\begin{array}{c|c} (U)_{ij} & (U)_{iJ} \\ \hline (U)_{Ij} & (U)_{IJ} \end{array}\right). \tag{2.13}$$

It is an orthogonal matrix, so that the column vectors have length 1 and are orthogonal to the row vectors. Moreover from the transformation (2.4) we find

$$\begin{pmatrix} (U)^{ij} \\ \hline (U)^{Ij} \end{pmatrix} \longrightarrow \left(\begin{array}{c|c} (e^{\epsilon^A T^A})^{ik} & (e^{\epsilon^A T^A})^{iK} \\ \hline (e^{\epsilon^A T^A})^{Ik} & (e^{\epsilon^A T^A})^{IK} \end{array}\right) \begin{pmatrix} (U)^{kl} \\ \hline (U)^{Kl} \end{pmatrix} \left(e^{-i\rho(\phi,\epsilon)}\right)^{lj},$$

$$e_a^i d\phi^a \longrightarrow e_a^l d\phi^a \left(e^{-i\rho(\phi,\epsilon)}\right)^{li}.$$

By using the Lie-variation we may write the second transformation in the infinitesimal form

$$\mathcal{L}_{R^A} e_a^i \equiv R^{Ab} e_{a,b}^i + R^{Ab}{}_{,a} e_b^i = -\rho(\phi, \epsilon)^{ij} e_a^j.$$

Together with $\mathcal{L}_{R^A} R^{Bb} = f^{ABC} R^{Cb}$, given by (2.6), these observations lead us to the following relation between the vielbein e_a^i and the Killing vectors R^{Aa}

$$R^{Aa} e_a^j \equiv \begin{pmatrix} R^{ia} e_a^j \\ \hline R^{Ia} e_a^j \end{pmatrix} = \begin{pmatrix} (U)^{ij} \\ \hline (U)^{Ij} \end{pmatrix}. \tag{2.14}$$

We note that

$$R^{Aa}\big|_{\phi=0} = \delta^{Aa}, \qquad e_a^i\big|_{\phi=0} = \delta_a^i, \tag{2.15}$$

by the construction. The relation should be understood as implying local equivalence of both hand sides at the origin $\phi^a = 0$.

The metric of the coset space may be naively given by

$$g^{ab} = R^{Aa} R^{Bb}, \qquad g_{ab} = e_a^i e_b^i. \tag{2.16}$$

However this way of giving the metric may be generalized if the broken generators X^i in (2.1) are decomposed as $\{X^{i_1}, X^{i_2}, \ldots, X^{i_n}\}$ under H, or equivalently the adjoint representation $\boldsymbol{N}^G$ is decomposed as

$$\boldsymbol{N}^G = \boldsymbol{N}^{i_1} \oplus \boldsymbol{N}^{i_2} \oplus \cdots \oplus \boldsymbol{N}^{i_n} \oplus \boldsymbol{N}^H \tag{2.17}$$

in the Cartan–Weyl basis. Here $\boldsymbol{N}^{i_r}$, $r = 1, 2, \ldots, n$, denote the N_r-dimensional representation of H, in which i_r stands for a set of the roots for the representation. Hence (2.13) becomes

$$\mathcal{D}(U) = (U)_{N_G \times N_G} = \left(\begin{array}{ccc|c} (U)_{N_1\times N_1} & \cdots & (U)_{N_1\times N_n} & (U)_{N_1\times N_H} \\ \vdots & \ddots & \vdots & \vdots \\ (U)_{N_n\times N_1} & \cdots & (U)_{N_n\times N_n} & (U)_{N_n\times N_H} \\ \hline (U)_{N_H\times N_1} & \cdots & (U)_{N_H\times N_n} & (U)_{N_H\times N_H} \end{array}\right), \tag{2.18}$$

with $N_G = \dim G$ and $N_H = \dim H$. Then the metric (2.16) can be generalized to

$$g^{ab} = R^{Aa}\left(UPU^{-1}\right)^{AB}R^{Bb}, \qquad g_{ab} = \eta^{ij}e_a^i e_b^j, \tag{2.19}$$

with a projection operator P such as

$$P = \left(\begin{array}{c|c} (P)_{ij} & (P)_{iJ} \\ \hline (P)_{Ij} & (P)_{IJ} \end{array}\right) = \left(\begin{array}{c|c} (\eta^{-1})_{ij} & 0 \\ \hline 0 & 0 \end{array}\right).$$

Here η^{ij} is given by

$$\eta^{ij} = \begin{pmatrix} (c_1)_{N_1\times N_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (c_n)_{N_n\times N_n} \end{pmatrix},$$

in which we have $(c_r)_{N_r\times N_r} = c_r(\mathbf{1})_{N_r\times N_r}$ with some constants c_r, $r = 1,\dots,n$ [11]. It is worth noting that the inverse of the vielbein e_a^i are given by

$$e^{ai} = R^{Aa}U^{Ai}.$$

3. The classical exchange algebra

We are now in a position to discuss the classical exchange algebra of the ordinary non-linear σ-model on G/H. The non-linear σ-model on G/H is given by

$$S = \int d^2\xi\ \mathcal{L} = \frac{1}{2}\int d^2\xi\ \eta^{\mu\nu}g_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b,$$

in the two-dimensional flat world-sheet. Here use is made of the formula (2.16) or (2.19) for the metric $g_{ab}(\phi)$. We study the Poisson structure on the light-like plane $x^+ = y^+$. The Dirac method hardly works to set up the Poisson brackets $\{\phi^a(x)\underset{,}{\otimes}\phi^b(y)\}$, because there appear the first and second class constraints, which we do not know how to disentangle. The reader may refer to [9] for a detailed argument on this. Hence we assume that the Poisson brackets can be set up so as to satisfy the two requirements. In the first place the energy–momentum tensor T_{--} should reproduce diffeomorphism as

$$\begin{aligned}\delta_{diff}\phi^a\left(x^+,x^-\right) &\equiv \epsilon\left(x^-\right)\partial_-\phi^a\left(x^+,x^-\right) \\ &= \int dy^-\epsilon\left(y^-\right)\left\{\phi^a(x), T_{--}\left(\phi(x)\right)\right\}\Big|_{x^+=y^+},\end{aligned} \tag{3.1}$$

by using the Poisson brackets $\{\phi^a(x)\underset{,}{\otimes}\phi^b(y)\}$. Secondly they should be consistent with the Jacobi identities. These requirements are satisfied with the Poisson brackets of the following form

$$\begin{aligned}\left\{\phi^a(x)\underset{,}{\otimes}\phi^b(y)\right\} = -\frac{1}{4}\big[&\theta(x-y)t^+_{AB}\delta^A\phi^a(x)\otimes\delta^B\phi^b(y) \\ &- \theta(y-x)t^+_{AB}\delta^A\phi^b(y)\otimes\delta^B\phi^a(x)\big]\end{aligned} \tag{3.2}$$

on the light-like plane $x^+ = y^+$. The notation is as follows. $\theta(x)$ is the step function. $\delta^A\phi^a(x)$ are the Killing vectors defined by (2.5). More correctly they should be written as $\delta^A\phi^a(\phi(x))$, but the dependence of $\phi(x)$ was omitted to avoid an unnecessary complication. The quantity t^+_{AB} is the most crucial in our arguments. It is a modified Killing metric which defines the classical r-matrices as

$$\begin{aligned} r^\pm &= \sum_{\alpha\in R}\operatorname{sgn}\alpha\, E_\alpha\otimes E_{-\alpha} \pm \sum_{A,B} t_{AB}T^A\otimes T^B \\ &\equiv t^\pm{}_{AB}T^A\otimes T^B, \end{aligned} \tag{3.3}$$

with T^A the generators of the group G given in the Cartan–Weyl basis as $\{E_{\pm\alpha}, H_\mu\}$, t_{AB} the corresponding Killing metric and $\operatorname{sgn}\alpha = \pm$ according as the roots are positive or negative. Note the relation $t^+_{AB} = -t^-_{BA}$. The r-matrix satisfies the classical Yang–Baxter equation (1.4) [12]. It is easy to show that the first requirement (3.1) is satisfied with the Poisson brackets (3.2). The second requirement can be similarly shown as has been done in [3,9]. (See Eqs. (3.20) and (12) in the respective reference.)

Note that

$$\left\{\phi^a(x)\underset{,}{\otimes}\Psi(y)\right\} = \left\{\phi^a(x)\underset{,}{\otimes}\phi^b(y)\right\}\frac{\delta\Psi(y)}{\delta\phi^b(y)}.$$

By using the Poisson brackets (3.2) we can easily show the Killing scalar defined by (2.9) to satisfy the classical exchange algebra in the form

$$\left\{\Psi(x)\underset{,}{\otimes}\Psi(y)\right\} = -\frac{1}{4}\left[\theta(x-y)r^+ + \theta(y-x)r^-\right]\Psi(x)\otimes\Psi(y), \tag{3.4}$$

on the light-like plane $x^+ = y^+$. Here $\Psi(x)$ should be understood with an abbreviated notation for $\Psi(\phi(x))$. It is also understood as generalized by means of (2.12), when the representation of G does not contain any singlet in the decomposition under H. Thus the classical exchange algebra (3.4) exists with Ψ in any representation of G.

4. Conclusions

Finally we would like to comment on the S-matrix for the PSU(2|2) spin-chain system found by Beisert [2]. The S-matrix is related to the R-matrix as $S = P\circ R$ with the permutation map P. Let us multiply P on the quantum exchange algebra (1.1) and the Yang–Baxter equation (1.2). Calculate the l.h.s. as $PR_{yx}P^{-1}P\Psi(y)\otimes\Psi(x)$. Then the respective formula becomes

$$S_{xy}\Psi(x)\otimes\Psi(y) = \Psi(y)\otimes\Psi(x), \qquad S_{yx}S_{zx}S_{zy} = S_{zy}S_{zx}S_{yx}.$$

In [2] the S-matrix was found by numerically solving the Yang–Baxter equation. Actually the solution had the extended symmetry $\mathrm{PSU}(2|2)\ltimes\mathbb{R}^3$. The classical r-matrix was discussed from this S-matrix with an appropriate deformation parameter in [13]. The resulting r-matrix is not of the kind which follows from the Poisson structure of some underlying theory for the spin-chain system.

We would like also to comment on the algebra appearing in the integrable non-linear σ-models on the symmetric space G/H or the principal chiral coset space $G\otimes G/G$ [14,15]. It reads

$$\left\{T(\lambda)\underset{,}{\otimes}T(\mu)\right\} = \left[r_{\lambda\mu}, T(\lambda)\otimes T(\mu)\right], \tag{4.1}$$

or some modified one due to the presence of a non-ultra-local term. Here $T(\lambda)$ is the Wilson line operator

$$T(\lambda) = P\exp\int_{-\infty}^{\infty} dx\, L(\lambda,x),$$

in which $L(\lambda,x)$ is a one-parameter family of a flat connection constructed from a conserved current $j_\mu(x)$ of the models [15]. The fundamental Poisson brackets are set up for $j_\mu(x)$ on the $t = const$ plane instead of the light-like plane. The Poisson bracket in (4.1) is calculated by using them. The reader may refer to [14] for the resulting r-matrix $r_{\lambda\mu}$ as well as more details on the arguments. We confine ourselves to remark that there appears no non-ultra-local term in calculating the Poisson brackets on the light-like plane. Hence the whole calculation in this Letter is free from the problem of a non-ultra-local term. The algebra (4.1) is a classical exchange algebra in a similar sense as (3.4), although the form of the r-matrix is different from (3.3). The similarity can be seen as follows. The corresponding algebra to (4.1) is obtained from (3.4) when considered with Ψ in the tensor product representation $\boldsymbol{N} \otimes \overline{\boldsymbol{N}}$ of G. Beyond this special representation the classical exchange algebra in the formulation on the $t = const$ plane has not been studied to the author's knowledge.

In this Letter we have studied the Poisson structure of a generic non-linear σ-model on G/H formulating on the light-like plane. Setting up the fundamental Poisson brackets (3.2) we have found the Killing scalar satisfying the classical exchange algebra (1.3) in an arbitrary representation, i.e., (3.4). It is natural to think of a certain non-linear σ-model with the PSU(2|2) or PSU(2, 2|4) symmetry as a underlying world-sheet theory for the spin-chain system [1,2]. We need a special care in order to generalize our arguments to the case where the symmetry admits non-trivial central extension as $\mathrm{PSU}(2|2) \ltimes \mathbb{R}^3$. The exchange algebra for such a non-linear σ-model will be discussed in a future publication.

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